The Black-Scholes Equation

Volatility

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The Black Scholes formula has often been described as using "the wrong number in the wrong formula to get the right price"

History

In 1973 Fischer Black, a finance contractor and Myron Scholes, an assistant professor of finance at MIT published a paper explaining the Black-Scholes Option Pricing Model. Almost all present day approaches and techniques used to estimate pricing are based on this Black-Scholes Model. Scholes received the Noble Prize in Economics in 1997, with an mention to Black because he had passed away in 1995.

Derivative

- Agreement between two parties that has value
- Value based on future price of underlying asset
- Asset = stock, bond, currencies, interest rates
- Price of asset can change
- Derivative agrees to price despite changes

What is the Black-Scholes Equation?

- Used in financial Mathematics
- Price or worth of a derivative
- Depends on changes in underlying assets
- Calculates large numbers of options prices in a short time
- Only calculates price at time of expiration
- European Vs. American options

Equation of a Portfolio

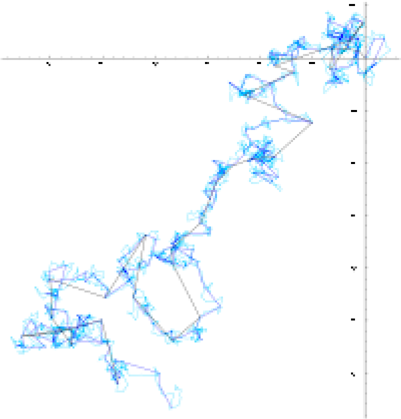
- Π = The portfolio
- S_t = Price of Stock
- Δ_t = number of shares owned at time t
- $P(S_t, t)$ = Price of the option
- Find values for dS_t and $dP(S_t, t)$

$$d\Pi = \Delta_t dS_t - dP(S_t, t) \quad (1)$$

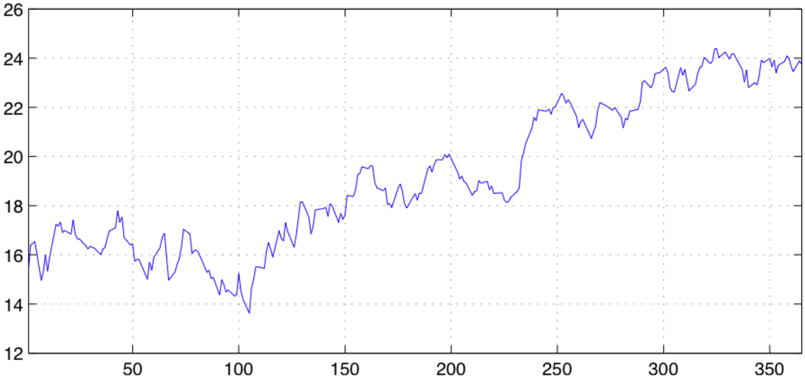
Geometric Brownian Motion

- One requirement for the Black-Scholes equation to hold is an assumption that the investment being made is risk free.
- Using the Equation for Geometric Brownian Morion the risk can be eliminated
- Stochastic Process
- Brownian Motion

Brownian Motion



Price of Stock



Equation for Geometric Brownian Motion

- $\mu =$ Rate at which value of stochastic process changes, annualized
- $\sigma =$ Volatility
- $dz_t =$ The risk

$$\mathrm{dS}_t = \mu S_t d_t + \sigma S_t dz_t \quad (2)$$

- Substitute this value into the portfolio equation
- Need to eliminate dz_t term in order to eliminate all risk

Ito's Lemma

- Ito's lemma is an extension of the chain rule for derivative
- Using it on $dP(S_t, t)$ forms the equation:

$$\mathrm{dP}(\mathrm{S}_t, t) = (\frac{\partial P(S_t,t)}{\partial S}\mu S_t + \frac{\partial P(S_t,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 P(S_t,t)}{\partial S^2}\sigma^2 S_t^2) \ (3)$$

Forming the Equation for the Derivative of the Portfolio

- Using (2) and (3) to substitute values for dS_t and $dP(S_t, t)$ the equation for the derivative of the portfolio is becomes:

$$\mathrm{d}\Pi = (\Delta_t \mu S_t - \frac{\partial P(S_t,t)}{\partial S} \mu S_t - \frac{\partial P(S_t.t)}{\partial t} - \frac{1}{2} \frac{\partial^2 P(S_t,t)}{\partial S^2} \sigma^2 S_t^2) dt + (\Delta_t \sigma S_t - \frac{\partial P(S_t,t)}{\partial S} \sigma S_t) dz_t \ (4)$$

- Use equation to find the number of shares owned at a certain time(Δ_t) which will eliminate the risk(dz_t):

$$\Delta_t = \frac{\partial P(S_t,t)}{(\partial S)} \ (5)$$

Risk-less Portfolio

- Substitute the value for Δ_t in equation for portfolio
- Eliminate Risk - Condition for Black-Scholes
- New Equation:

$$\mathrm{d}\Pi = \left(-\frac{\partial P(S_t,t)}{\partial S} S_t - P(S_t,t)\right)dt \quad (6)$$

Adding Interest Rate

- The Difference between a portfolio with a risk free interest rate and the one we just formed should be zero:
- Set $d\Pi = r\Pi dt$
- The Black-Scholes PDE is formed:

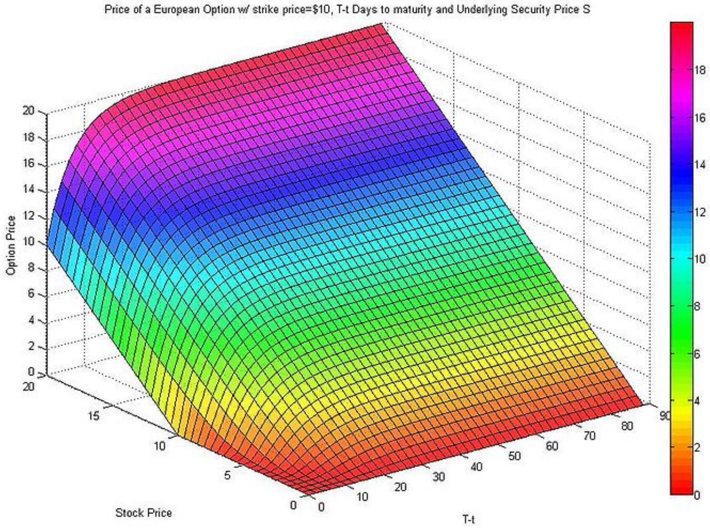
$$rP(S_t, t) = \frac{\partial P(S_t,t)}{\partial t} + rS_t \frac{\partial P(S_t,t)}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 P(S_t,t)}{\partial S^2} \quad (7)$$

Volatility

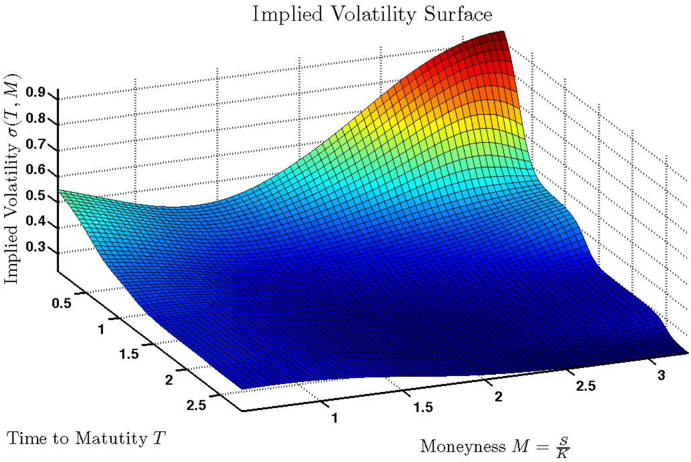
- Measure of risk based on standard deviation of the value of an asset over a specific time
- Must be estimated
- Implied - Price of an option
- Historical - Value of an option
- High/Low Volatility
- Use constant volatility to solve equation

Constant Volatility

Black-Scholes assumes constant Volatility



Volatility Smile



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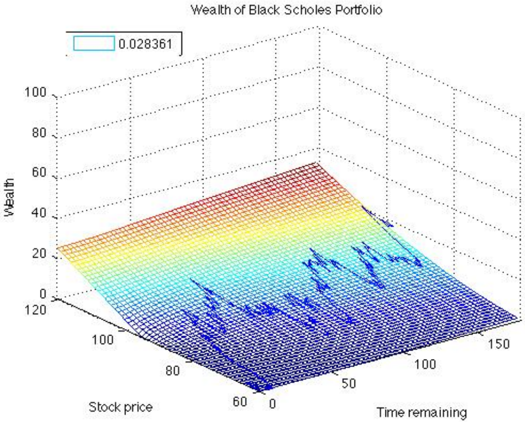
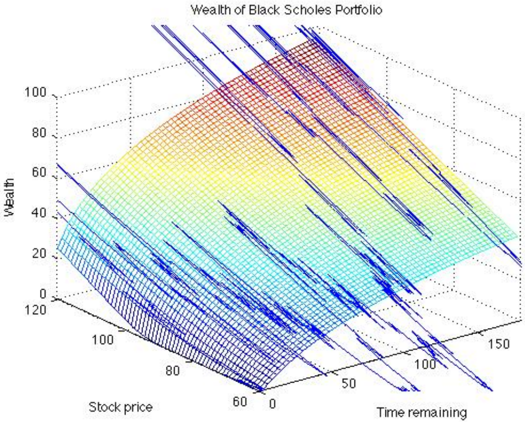
Low Volatility



High Volatility



Future Work

- Use equation to compute volatility instead of random
- Compare these prices to historical volatility
- Black Scholes pricing model using changing volatility over time
- What is more accurate?

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