The Black-Scholes Equation and Volatility

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The Black Scholes formula has often been described as using ”the wrong number in the wrong formula to get the right price”

Abstract

The Black-Scholes equation is a commonly used model in financial mathematics which allows one to analyze the price or worth of a derivative over time and changes in underlying assets. A major issue with equation is that it typically assumes a more stable volatility than what actually is. In this paper, I will discuss the formation of the Black-Scholes equation and numerical simulations of the Black-Scholes partial differential equation, using different values for volatility.

1 History

In 1973 Fischer Black, a finance contractor and Myron Scholes, an assistant professor of finance at MIT published a paper explaining the Black-Scholes Option Pricing Model. Almost all present day approaches and techniques used to estimate pricing are based on this Black-Scholes Model. Scholes received the Noble Prize in Economics in 1997, with an mention to Black because he had passed away in 1995.

2 Derivative

A derivative is an agreement between two parties that has a value. This value is based on the price of an underlying asset, based on it’s past behavior and future predictions. The underlying asset can be anything that has a value such as a stock, bond, currency or interest rate. The values of these assets will change but the agreed value of the derivative will not.

3 Option

An option is a derivative that creates an agreement between two parties on a price and amount of time the parties have to buy/sell the asset. The buyer may by this option up until the expiration date at the agreed price, but does not have to. On the other hand the seller must sell at any time the buyer decides to exercise his right to buy.

3.1 American Option

May be exercised any date before or on the expiration date. This makes the Black-Scholes Equation less accurate in price prediction because it only predicts the price at the time of expiration.

3.2 European Option

May only be exercised on the expiration date. The Black-Scholes equation can predict the pricing of a European option with better accuracy because of the conditions to only purchase on the expiration date.

4 The Black-Scholes Equation

The Black-Scholes Equation is an equation used in financial mathematics. It calculates the price or worth of a derivative over a certain amount of time. The price represents the price at the time of expiration of the option, but not at anytime before. This allows the equation to calculate large numbers of option prices in a short amount of time. This is also one of the major flaws of the equation considering in the United States one can exercise the right to buy an option at anytime until and including the expiration date. Another major flaw of the Black-Scholes equation is the use of constant volatility. Volatility is the measure of risk based on the standard deviation of the value of an asset over a specific time, in this case until the expiration. In reality the value of an asset will change randomly, not with a specific constant pattern. Despite these flaws the Black-Scholes equation is an important tool in finance.

5 Portfolio

A portfolio is the collection of investments held by a institution or individual. These investments include all types of assets from a bank account to bond. The purpose of holding a portfolio is to minimize the risk of owning many assets at one time.

$$d\Pi = \Delta_t dS_t - dP(S_t, t) \tag{1}$$

- Π = The portfolio
- S_t = Price of Stock
- Δ_t = number of shares owned at time t
- $P(S_t, t)$ = Price of the option

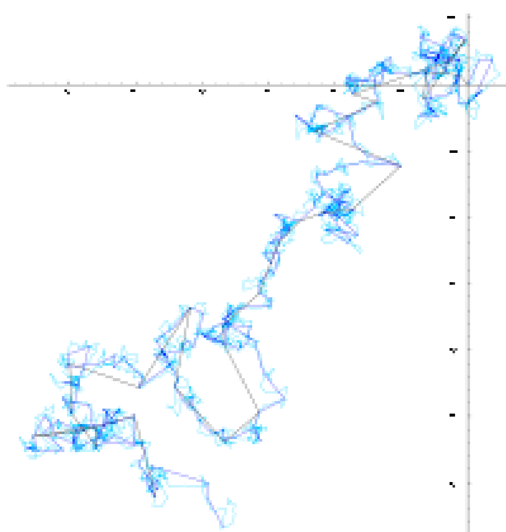
This equation represents the derivative of the equation of a portfolio, or the change in the value of the portfolio over time. By finding values for dS_t, the change in the price of the stock over time and $dP(S_t, t)$, the change in the price of the option over time. An equation for the change in the value of the portfolio can be formed. Using and equation for Geometric Brownian Motion and Ito's Lemma values are found.

6 Geometric Brownian Motion

One requirement for the Black-Scholes equation to hold is an assumption that the investment being made is risk free. A risk free investment is one that has a value that does not vary much from the original value over the amount of time it is being held. Using Geometric Brownian Motion the risk can be eliminated.

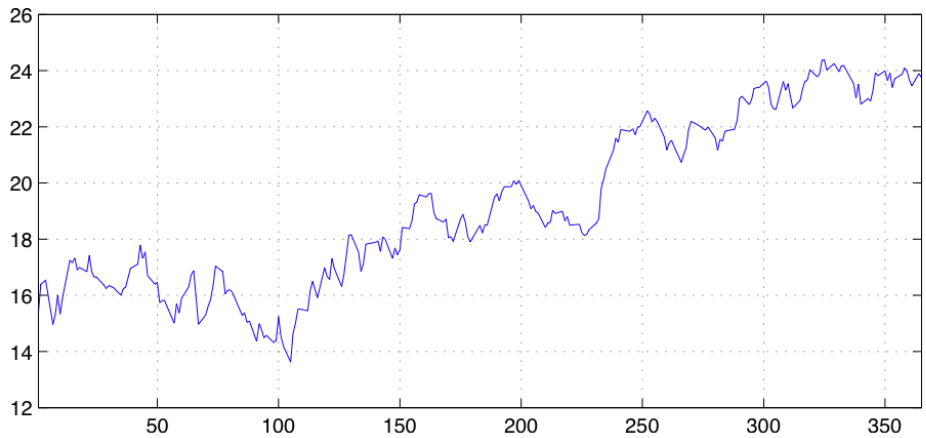
6.1 Brownian Motion

Brownian Motion is a stochastic process. A stochastic process is one that happens at random, there is no real prediction for the behavior of it over time. The definition of Brownian Motion is the prediction that particles suspended in fluid will move at random. This arbitrary pattern is considered similar to the way prices of stocks in the stock market change over time.



Brownian Motion

This is a graph of Brownian Motion: the random movement of a particle in fluid.



Price of Cisco Stock

This graph is an example of a stock price changing over time. It looks similar to the graph of Brownian Motion but a little less sporadic. There is no accurate way to predict an exact stock price, Brownian Motion is used as component the Black-Scholes equation to show this.

6.2 Equation for Geometric Brownian Motion

$$dS_t = \mu S_t d_t + \sigma S_t dz_t \tag{2}$$

- μ = Rate at which value of stochastic process changes, annualized
- σ = Volatility
- dz_t = The risk

This equation represents the change in the price of the stock over a certain amount of time: dS_t. This is one of the variables needed in the equation for the derivative of the portfolio. This equation also introduces the risk term: dz_t. Eventually by eliminating this term the portfolio will be risk free, a condition for the Black-Scholes equation.

7 Ito's Lemma

Ito's lemma is part of Ito calculus which allowes the of use calculus in stochastic processes. Ito calculus can be used to find integrals but in this case a part of Itos calculus, Ito's lemma, will be used to find the derivative of $P(S_t, t)$. Ito's lemma is the stochastic calculus version of the chain rule, using the taylor series expansion. It is commonly used in financial mathematics, specifically in the derivation of the Black-Scholes equation.

Using Ito's Lemma on $dP(S_t, t)$ forms the equation:

$$dP(S_t, t) = (\frac{\partial P(S_t, t)}{\partial S}\mu S_t + \frac{\partial P(S_t, t)}{\partial t} + \frac{1}{2}\frac{\partial^2 P(S_t, t)}{\partial S^2}\sigma^2 S_t^2) \tag{3}$$

8 Equation for the Derivative of a Portfolio

Geometric Brownian Motion found a value for the derivative of the stock price and using Ito's lemma a value for the price of the option was found. By substituting these values into the equation for the derivative of the portfolio a new equation is formed.

$$d\Pi = (\Delta_t \mu S_t - \frac{\partial P(S_t, t)}{\partial S} \mu S_t - \frac{\partial P(S_t, t)}{\partial t} - \frac{1}{2} \frac{\partial^2 P(S_t, t)}{\partial S^2} \sigma^2 S_t^2) dt + (\Delta_t \sigma S_t - \frac{\partial P(S_t, t)}{\partial S} \sigma S_t) dz_t \quad (4)$$

Now the dz_t or the risk term is included in the equation. The Black Scholes equation requires that there be no risk, so this term must be eliminated by finding a Δ_t or a number of shared owned at time t that gets rid of the risk. The number of shares which does this is:

$$\Delta_t = \frac{\partial P(S_t, t)}{(\partial S)} \quad (5)$$

9 Risk Free Portfolio

The definition of a risk free portfolio is one that does not change in value over the time it is held. By eliminating the dz_t term the risk is eliminated but a interest rate must still be factored in. To do that we must take:

$$d\Pi = (-\frac{\partial P(S_t, t)}{\partial S} S_t - P(S_t, t)) dt \quad (6)$$

and set it equal to the equation with a risk free interest rate r. By setting $d\Pi = r\Pi dt$ the new equation with an interest rate is created.

$$rP(S_t, t) = \frac{\partial P(S_t, t)}{\partial t} + rS_t \frac{\partial P(S_t, t)}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 P(S_t, t)}{\partial S^2} \quad (7)$$

This is the Black-Scholes partial differential equation.

10 Volatility

"It (volatility) is only a good measure of risk if you feel that being rich then being poor is the same as being poor then rich" - Peter Carr(Managing Director at Morgan Stanley)

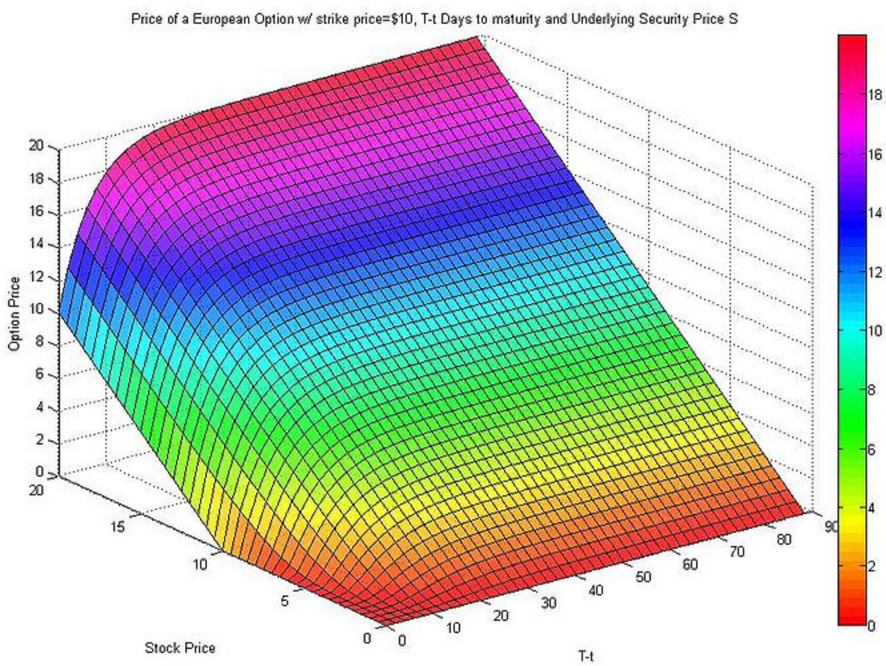
Volatility is the variation of the price of an asset over a certain amount of time, the time until expiration. The Volatility does not measure whether the prices increase or decrease it merely shows how large or small these price changes were or can be expected to be. The Black Scholes model assumes this value to be constant, which is not accurate to real markets. In reality volatility changes from high to low many times until expiration. An asset experiences high volatility when the prices move up and down frequently. Low volatility is when the prices do not move much over time.

10.1 Implied Volatility

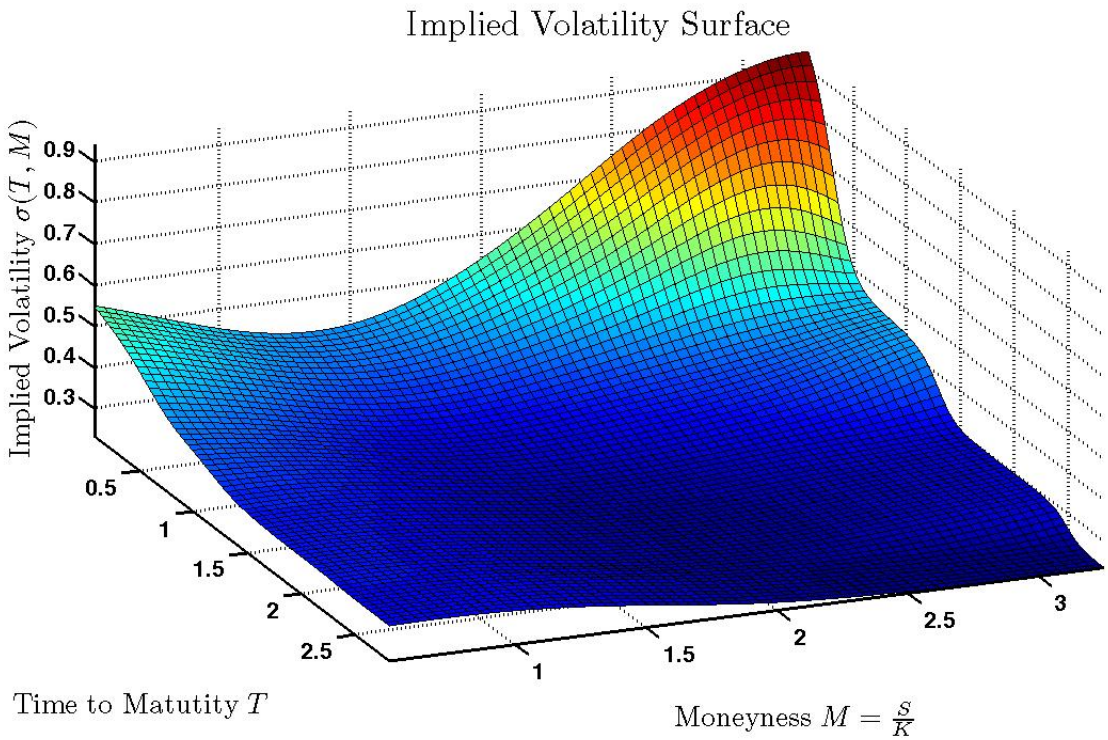
Implied volatility is volatility calculated from the price of a derivative. It is used in pricing to predict a price for a derivative using current, past or future prices.

10.2 Historical Volatility

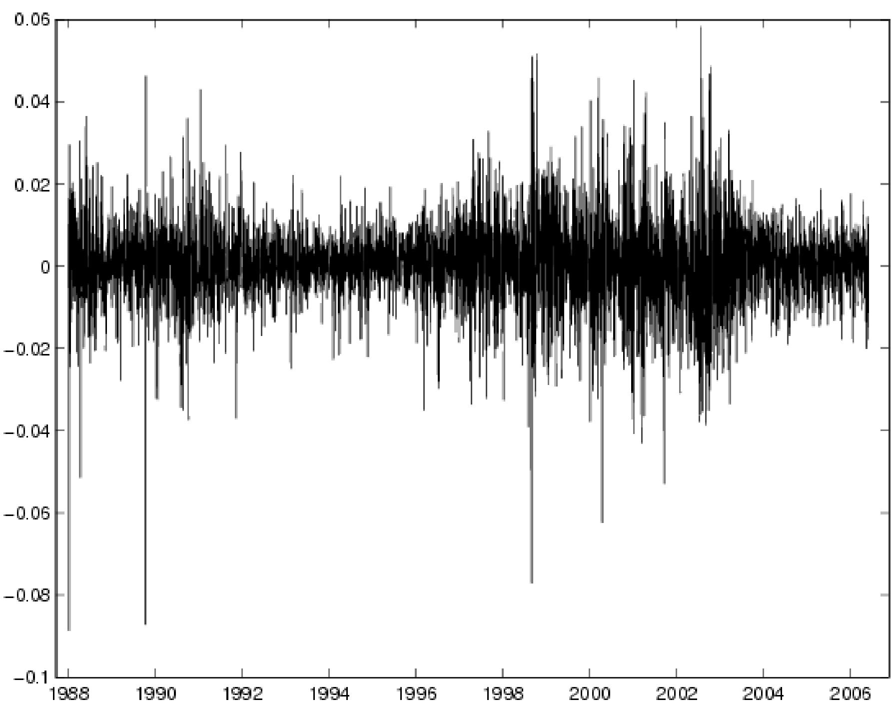
Historical Volatility is a prediction of the value of a derivative using prices from the past.



This is an example of a Black Scholes option pricing model using constant volatility. The graph has a constant slope which is not accurate to a volatility surface representing a real market.



This graph shows what a volatility surface should look like, the shape of a smile or skew



This is an example of how volatility changes over time. The jumps are frequent but rarely go over .03. This is very different than what a Black-Scholes plot of volatility would look like, a straight line at a certain y value. Although the difference is small, there is a difference and when investing hard earned money, it is important to use a value accurate to real markets.

11 Code

```
clear all; close all; clc

r     = log(1.04)/365;         % bond growth rate
mu    = log(1.05)/365;         %Stock price growth rate
Sigma = .01;    %Stock price volatility, using random values
init  = 100;                   % initial stock and bond price
c     = 95;                    % exercise/strike price

N=500;               % number of steps to take
T=180;               % expiration time
h=T/N;               % time step
t=(0:h:T);           % t is the vector [0 1h 2h 3h ... Nh]

% Initial values
```

Finding initial values for the equation for the price of an option, number of shares of stock and number of shares of a bond

```
  b(1)=init;          % initial bond price B_1
  p(1)=init;          % initial stock price S_1
    s = T;             % time argument - Changing variable T to s
  d1 = (log(p(1)/c)+(r+(sigma^2)/2)*s) ./ (sigma*sqrt(s));
  d2 = d1 - sigma*sqrt(s);
```

d_1 and d_2 are variables used in solving the Black-Scholes equation

```
  x(1) = p(1) .* normcdf(d1) - c*exp(-r*s) .* normcdf(d2); %price of option
```

$x(1)$ is equal to the equation for the price of an option, also represented as $C(S_t, t)$ where..

$$C(S_t, t) = N(d_1)S_1 - N(d_2)Ke^{-r(T-t)} \tag{8}$$

- $p(1) = S_1 =$ the initial stock price
- normcdf $= N =$normal cumulative distribution function. The cumulative distribution function of the normal distribution. The normal distribution is a probability distribution used to describe random variables which cluster to a certain mean value. The cumulative distribution function shows the probability that a random variable will be less than this value.

- $N(d_2)$ - Probability that the stock price will be above K
- $K = c =$ Strike price, agreed price of derivative
- $r = r =$ risk free interest rate
- $T - t = s =$ time to expiration

```
y(1) = x(1);
n(1) = normcdf(d1)+(normpdf(d1)-c*exp(-r*s)*normpdf(d2)/p(1))
/(sigma*sqrt(s)); %shares of stock
```

$$\text{Shares of Stock } = Nd_1 + \frac{(N_p(d_1) - Ke^{-r(T-t)} * \frac{N_p(d_2)}{S_1})}{\sigma(\sqrt{T-t})} \tag{9}$$

- normpdf= N_p = Normal probability density function, probability of this random variable occurring at a given point.

```
m(1) = (x(1)-n(1)*p(1))/b(1)  % shares of bond
```

$$\text{Shares of bond} = \frac{(C(S_1,t) - Nd_1 + \frac{(N_p(d_1) - Ke^{-r(T-t)} * \frac{N_p(d_2)}{S_1})}{\sigma(\sqrt{T-t})} * S_1)}{B_1} \tag{10}$$

m(1)= shares of the bond = (Price of the option - shares of stock * initial stock price) / the initial bond price.
Finding values for the functions that change over time by taking steps:

```
for i=1:N           % start taking steps
  b(i+1)=b(i)+r*b(i); % bond price:
```

Bond price + risk free interest rate multiplied by bond price = total bond price, bond price after interest rate is included over time... summed from $i = 1 : N$. Adding the interest rate to the solution.

```
  p(i+1)=p(i)+mu*p(i)*h+sigma*p(i)*sqrt(h)*randn; %stock price
```

Stock Price using i from 1 to N steps...

$$p(i+1) = S_i + \mu S_i h + \sigma S_i \sqrt{(}h)(randn) \tag{11}$$

- μ is the Drift rate or the change in the average value of the stochastic process

- h is the time step T/N... T= expiration time and N = number of steps
- σ is the volatility
- randn= gives pseudorandom values drawn from the standard normal distribution

```
y(i+1) = y(i) + n(i)*(p(i+1)-p(i)) + m(i)*(b(i+1)-b(i));  % wealth
```

- This represents how much the options price will grow or shrink, the wealth over time.
- By adding the change in price of the stock and the change in price of the bond
- Finding the change in price of option by adding the changes of stock price and bond price
- Using Newton's method to find the change in value or derivative of the stock price and bond price over values of i from 1 to N.

```
s = T-t(i+1);                % time argument using steps

d1 = (log(p(i+1)/c)+(r+(sigma^2)/2)*s) ./ (sigma*sqrt(s));
d2 = d1 - sigma*sqrt(s);
```

d_1 and d_2 using $p(i+1)$ instead of $p(1)$, used in taking steps
Use changing time, s and $i+1$ in the original equations to get change in shares of stocks and bonds, and price of the option over time. Re-stating values of $x(i)$, $n(i)$, $m(i)$, using the steps from $i = 1 : N$

```
x(i+1) = p(i+1) .* normcdf(d1) - c*exp(-r*s) .* normcdf(d2);
%price of option
```

Equation for the price of an option using time steps, and a changing initial value for the stock price.

```
n(i+1) = normcdf(d1) + (normpdf(d1)-c*exp(-r*s)*normpdf(d2)/p(i+1))
/(sigma*sqrt(s));  %Shares of stock
```

Equation for the shares of the stock using time steps and change in initial prices over time.

```
m(i+1) = (x(i+1)-n(i+1)*p(i+1))/b(i+1);          % shares of bond
```

(Price of the option over N steps - shares of the stock over N steps* Price of stock over N steps) / Bond price = the number of shares of a bond

```
end;
```

Graphing surface and grid. Use tt, xx and uu to make surface:

```
[xx,tt] = meshgrid(60:1:120,0.05:3:180);     % prepare points on a grid
```

```
gg = (log(xx/c)+(r+(sigma^2)/2)*tt) ./ (sigma*sqrt(tt));
```

$$gg = \frac{(log(\frac{xx}{k}) + (r + \frac{\sigma^2}{2})tt)}{\sigma\sqrt(tt)} \tag{12}$$

Taking d_1 and replacing the variable for stock price with xx and the variable for time to tt

```
uu = xx .* normcdf(gg) - c*exp(-r*tt) .* normcdf(gg - sigma*sqrt(tt));
```

$$uu = xxN(gg) - Ke^{-rtt}N(gg - \sigma\sqrt(tt) \tag{13}$$

This is the equation for the price of an option using gg in place of d_1 and again replacing the variable for stock price with xx and the time argument with tt
uu becomes the 3-D surface, showing the change in price of the option.
Then creating a graph, label and setting axis' :

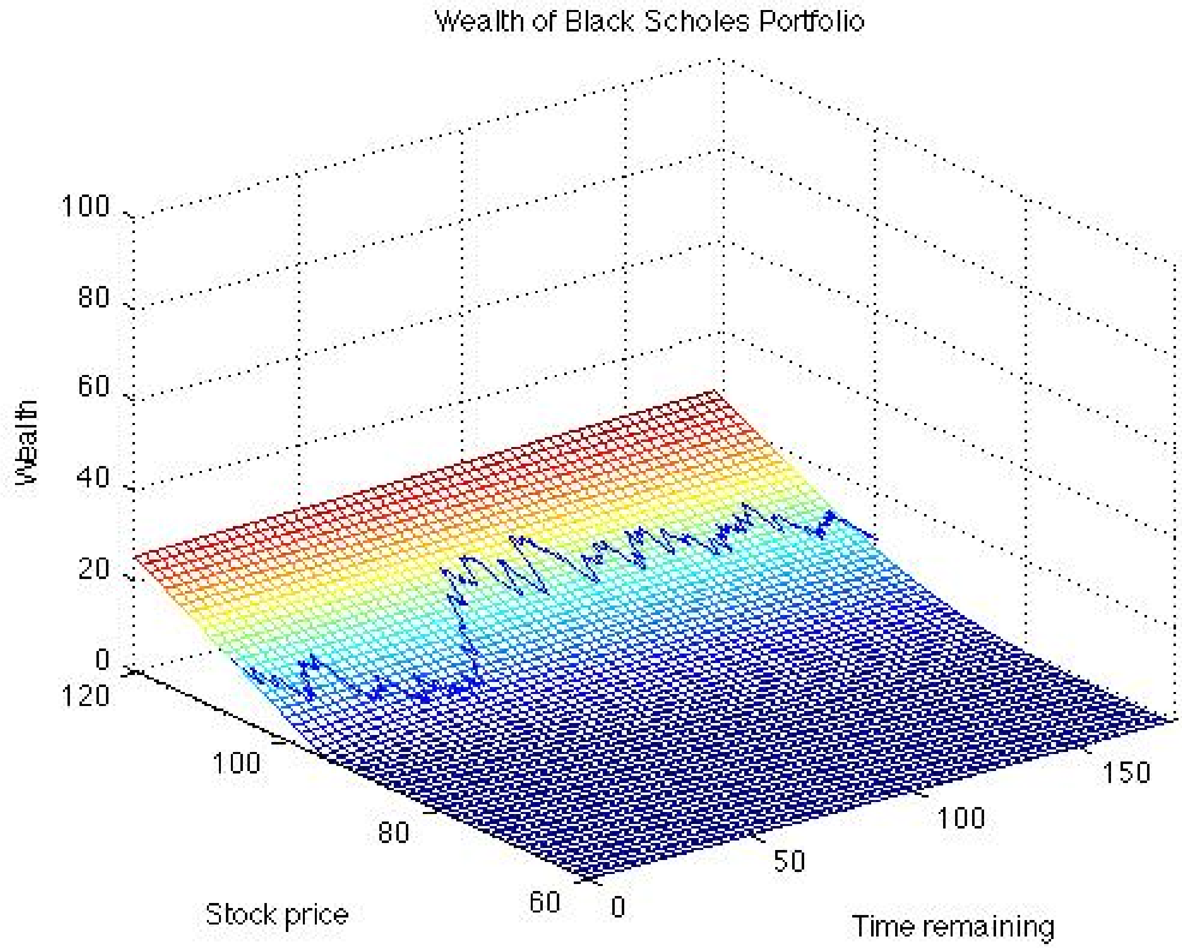
```
mesh(tt,xx,uu);
shading faceted;
Labeling
xlabel('Time remaining');
ylabel('Stock price');
zlabel('Wealth');
title('Wealth of Black Scholes Portfolio');
axis([0 180 60 120 0 100]); %Set size of axis
grid on;
hold on;


plot3(T-t, p, x+0.1);
```

Plot line in 3d space: price of an option of time until expiration in steps of x +.01....blue line

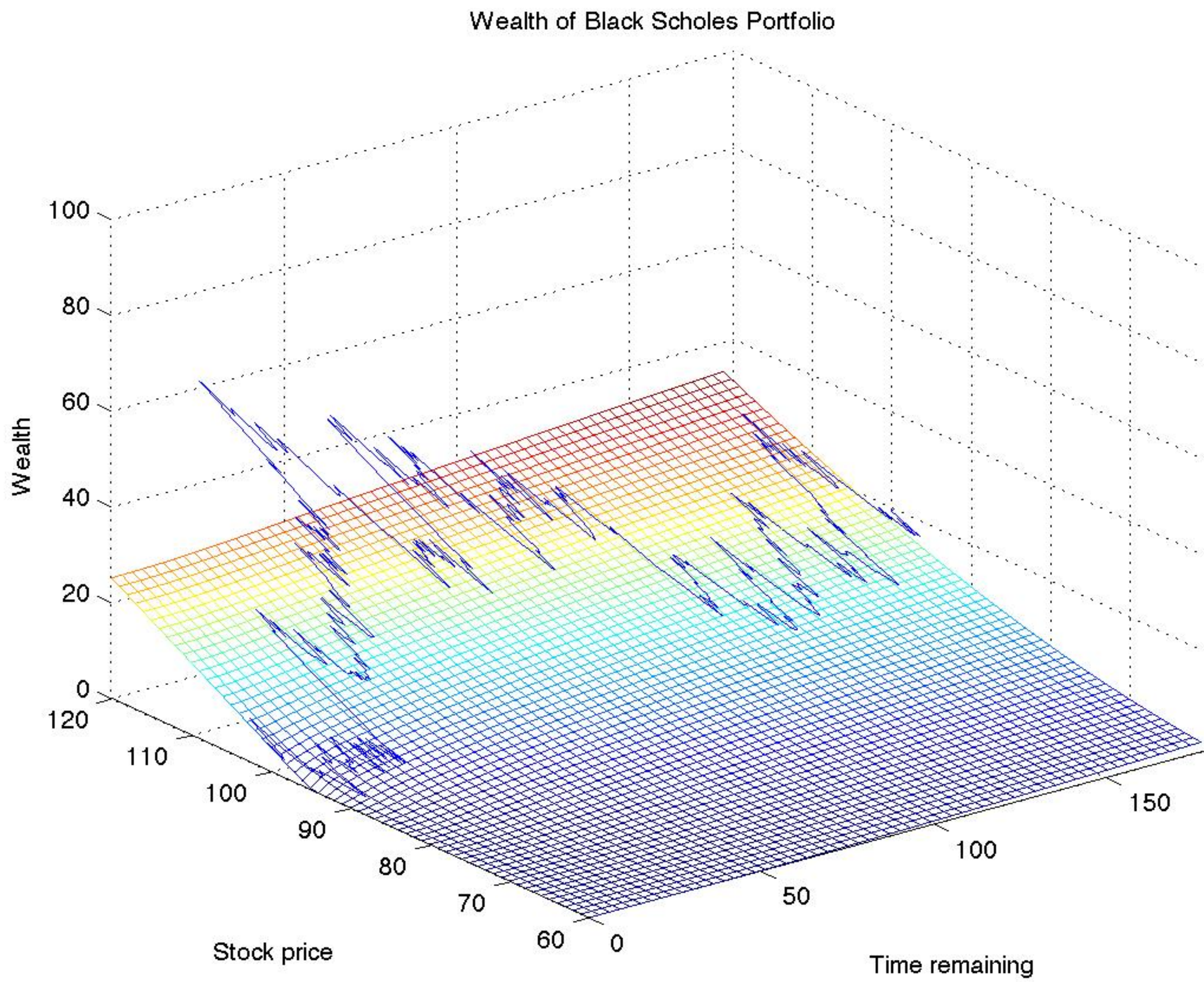
```
pause
clf
end
```

12 Low Volatility



The surface has a somewhat constant slope signifying a remotely constant change in stock prices over time. The stock prices have semi-regular jumps from high to low over the 180 days because the volatility is set at a small value, .01.

13 High Volatility



The volatility surface shown here has a greater slope due to the larger volatility. The stock prices are more erratic because of the size of the volatility which is set at .03.

14 Personal Reflection

I learned a lot about financial mathematics this semester. There were a few problems I ran into which slowed me down and didn't allow me to accomplish all I hoped to. The first set back was actually understanding the Black-Scholes equation. I had underestimated how complicated just figuring out what the equation did and how it was formed would be. I was hoping to be able to understand the solution and create a code of my own rather than having to alter someone else's. With that, I did learn in depth how the Black-Scholes equation is formed and actually understand it to a point where I could probably explain it to someone. That I think is an accomplishment in itself. I'm honestly more interested in pure mathematics than computational so I was hoping to understand how to solve the equation without using a code. I tried for a while to do this, but sort of gave up in order to focus more on volatility and why it is important to the Black-Scholes equation. I hope to explore the solution more in depth in the future. My original goal in all this was to find a better way to computationally compute volatility in the Black-Scholes equation. I didn't realize in the beginning that there were many different models for volatility and it wasn't just a somewhat simple equation. Volatility became a whole new "Black-Scholes equation" to me. While trying to produce different graphs and models of the Black-Scholes equation and volatility I ran into the issue of what to compare my future results to. There is no way to find a completely accurate price or volatility until it is observed. In the future I think I will compare my results to values form the past in order to understand the error. Because of my lack of skill in MATLAB I wasn't able to incorporate that equation into my code, in order to make a changing volatility. That I will consider my next goal in CSUMS. Overall I was not prepared for how complex and new financial mathematics would be to me. There are many terms and processes on must understand before getting into the computation. I have to say I learned a lot about this area of mathematics and am very interested in learning more. I hope to expand my ability to alter codes and hopefully create my own from scratch using the Heston model for volatility. I Though out this semester LATEX has become second nature to me, even though I always have so many errors I can produce a semi-goodlooking paper or beamer presentation as if I was typing something up on WORD. MATLAB has always been a challenge for me, something in my brain just doesn't get that and I didn't progress as much in this area as I would have liked to this semester because of all the time spent just building the equation. Another challenge which does not necessarily have to do with math itself is my fear of public speaking. Even though I'm still horrified at getting up in front of just a couple people, I got some experience under my belt. Despite there not being many people at Amherst when I presented I went there intending to talk in front of as many people showed and succeeded in that goal. I look back and even though I didn't accomplish everything I would have liked to I feel as if I improved my skills in many areas and found a topic I am very interested in continuing researching.

15 CSUMS Review

CSUMS is a good experience which allowed me to explore something I am actually interested in. The first semester I was given a topic which I thought was a very good strategy. Coming into CSUMS I was somewhat scared that I wouldn't come up with a research topic and really had no idea how to do research in math. The fact that a paper was given to me and I could just dissect it and try to understand the little pieces which seemed impossible was fun to me. Eventually I gained an understand of fourier series and with my partner created a code, this gave me confidence for next semester. This semester I was able to pick my own research topic which is one thing I like about CSUMS. In every other class you are told what you have to learn, study and memorize for a test. Here I could actually enjoy the topic I was learning about and pick and choose what pieces I'd like to learn and what I would save for later.

16 References

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17 Acknoledgements

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