Hybrid Trigonometric Polynomial Approximation

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Abstract

Fourier Series approximations are well known for their spectral convergence of data reconstructions on smooth and periodic functions. However, they fail to produce similar convergence when faced with discontinuous problems due to peculiar behavior near the discontinuities. Our work is to remedy this problem by using a hybrid method. In this method, polynomial approximation is used near the discontinuity and Fourier approximations are used on the other regions. We present numerical differences between our methods and other previous methods applied to similar popular problems.

Introduction

The goal of the project is to construct accurate point values of an unknown function $f(x)$ on $-1 \leq x \leq 1$. Given the first $2N+1$ coefficients, the Fourier Partial Sum

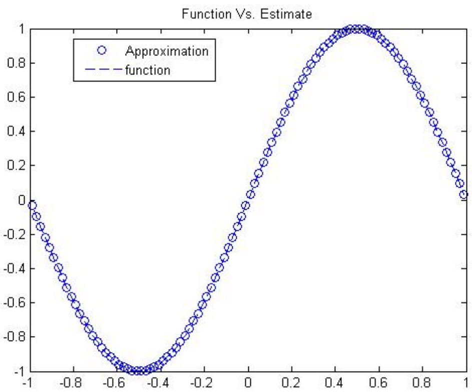
$$
\begin{aligned}
F_N(x) &= \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(\pi k x) + b_k \sin(\pi k x) \\
a_k &= \int_{-1}^{1} f(x) \cos(\pi k x) dx \\
b_k &= \int_{-1}^{1} f(x) \sin(\pi k x) dx
\end{aligned}
$$

Summing up the straight forward Fourier Series to construct an approximation is a good accurate reconstruction given that $f(x)$ is smooth and periodic:

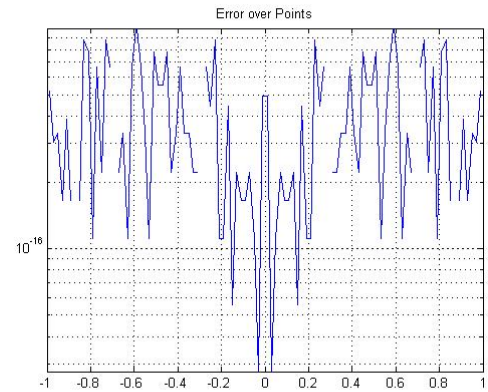
$$max_{a \leq x \leq b} |f(x) - F_N(x)| \leq e^{-N}$$

the approximation converges spectrally, the error decays exponentially. However, when the data is either not smooth, or not periodic, we have oscillations introduced into the reconstructions which then gives us polluted results.

Above in the pictures you can see the difference in the approximations when the only difference is that the data in one is smooth and periodic, while the other has a jump discontinuity at the boundaries. In the graph you can see the oscillations and happening near the boundaries and how it has an effect on the errors. This behaviour is known as *Gibbs Phenomenon*. Our research this semester was looking into recovering the accuracy of the Fourier reconstructions with data that would normally be polluted by the *Gibbs Phenomenon*.



(a)



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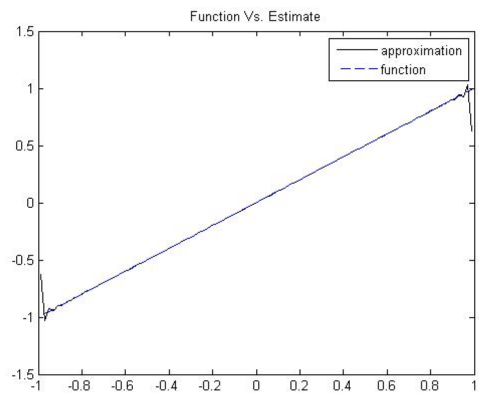
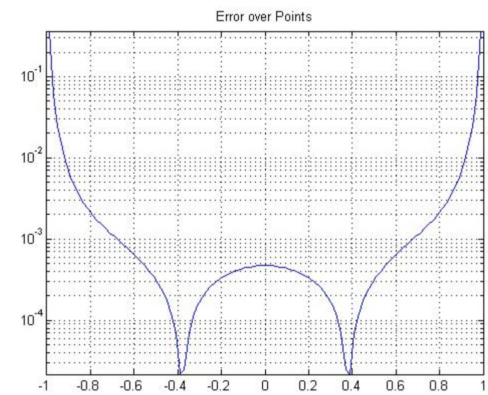


(c)



(d)

Figure 1: Parts (a) approximation of $f(x) = \sin(\pi x)$ (b) the error plot, log scale, of the approximation of $f(x) = \sin(\pi x)$ (c)approximation of $f(x) = x$ (d) the error plot, log scale, of the approximation of $f(x) = x$

Complex Exponential Fourier Series

Before we go into the different types of methods used this semester, we would like to introduce another equivalent representation of the Fourier Sum. Using Euler's polar Identity of

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we can represent the Fourier sums through complex exponentials. Above one can see that we have in order to calculate the partial sum we need to find to sets of coefficients, the sine and the cosine, and then pair them up with the corresponding sine/cosine term. This can lead to a large number of calculations given a high number of points, or a large number of coefficients. One way to reduce calculations is by using Euler's polar identity to re-represent the sum as

$$F_N(x) = \sum_{k=-N}^{N} c_n e^{ikx}$$
$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$$

We will be using this representation of the Fourier series throughout the rest of the report.

Window Functions

As if one were looking through a window, a person cannot see the whole world from the window but only see what the window is looking on to. This same idea applies with window functions. The user, or person working with the data, can apply certain operations on to the data so they may look at it pieces at a time, and as long as they eventually apply the inverse operations then the data should still be usable.

Using the following idea:

$$f(x) = \frac{f(x) * w(x)}{w(x)}$$
$$F_N(x) \approx \sum_{k=-N}^{N} c_k e^{ikx}$$

We multiply the given data points of the function f(x) by a function $w(x)$ to be reconstructed by a continuous function on the domain [0, 2] as done in [6].

$$\hat{f_k^w} = \frac{1}{2\pi}\int_0^{2\pi} w(x) * f(x)e^{-ikx}dx$$
$$F_N^w(x) \approx \frac{1}{w(x)}\sum_{k=-N}^{N} \hat{f_k^w} e^{ikx}$$

The function $w(x)$ must be a continuous function satisfying the following condition

$$w(x) = \begin{cases} 1 & : x \in [a,b] \subset [0,2\pi] \\ \epsilon & : x \notin [a,b] \end{cases}$$

We use ϵ to avoid division by 0. In practice, $\epsilon \approx eps$, machine precision. On $[0, 2\pi]$, we use

$$w(x) = e^{-\frac{(x-\pi)}{\pi}^{2\lambda}}$$

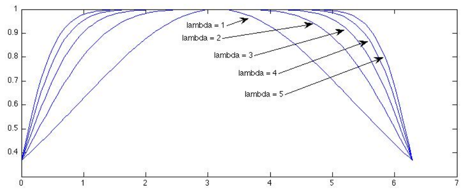
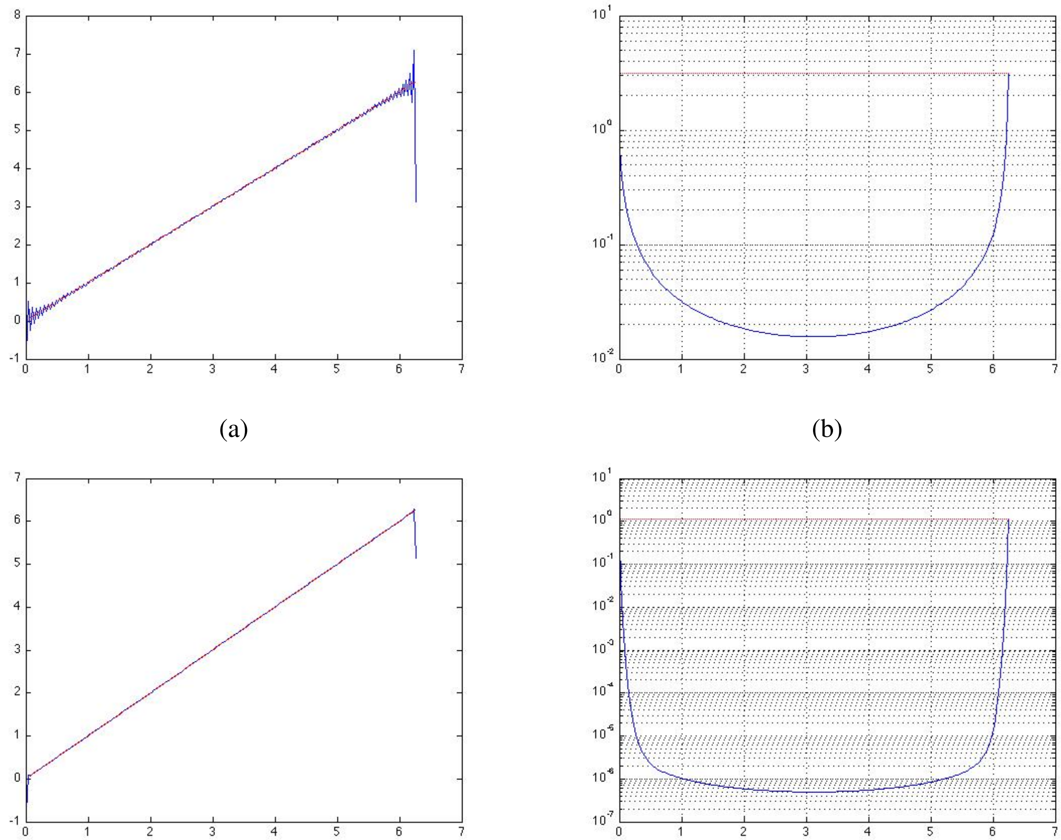


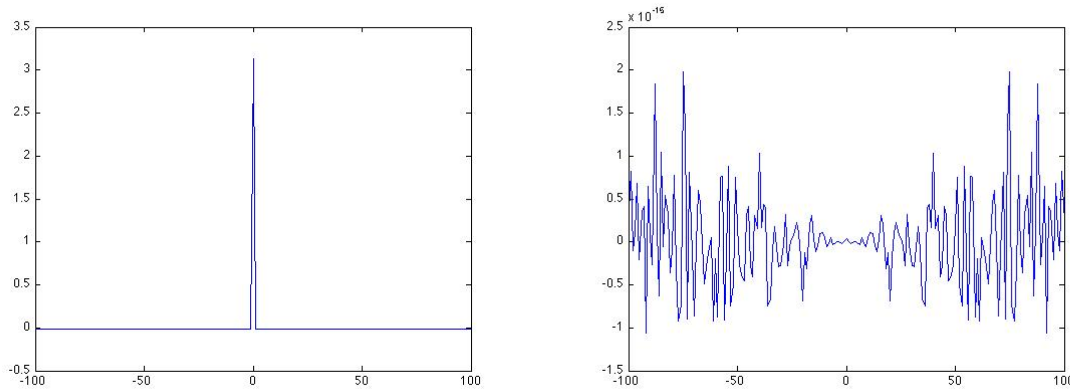
Figure 2: $w(x)$ for different lamdba ($\lambda = 1, 2, 3, 4, 5$)



(a) (b)

(c) (d)

Figure 3: Parts (a)$f(x) = x$ with $np = 201$ and $nc = 201$ (b) the error plot, log scale, $f(x) = x$ with $np = 201$ and $nc = 201$ (c)$f(x) = x$ with $np = 201$ and $nc = 201$ (d) the error plot, log scale, $f(x) = x$ with $np = 201$ and $nc = 201$



(a) (b)

Figure 4: Parts (a) coefficient of $f(x) = x$ (b) coefficient of $f(x) = sin(x)$

The following figures are comparison between this new method and the straight forward Fourier Series

Above you can see by simply focusing on the smooth regions of the domain we are able to gain back 5 times more accurate approximations. Eventhough the maximum errors seem to be the same, the overall all error shrunk drastically.This idea allowed us to minimize the global damage normally caused by the gibbs phenomenon by regarding the shock as less important information.

Filtering

In this section, we use the Fourier Series complex form

There are two stages for providing information about a function [3]:

- *Storage*: store the expansion coefficients
- *Retrieval*: Sum up the expansion

For the case of the discontinuous problem, it is unwise to simply sum up the coefficients. This is because they contain data polluted by the discontinuities and this information will jeopardize the reconstruction.

As described in [3, 4, 2], the coefficient $\hat{f}_k$ decays at a faster rate for continuous/periodic problems than the cases where *Gibbs Phenomenon* is present.

The following is shown in [5]:

$$\text{if} \sum_{|k|\leq\infty} |\hat{f}_k|^2 < \infty \rightarrow \text{then } ||f(x) - F_N||_{L^2_{[0,2\pi]}} \rightarrow 0 \quad (1)$$

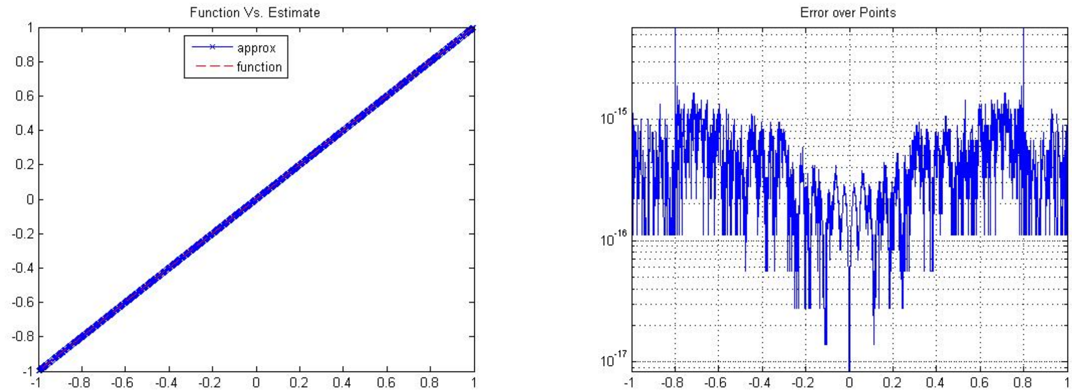
$$\text{if} \sum_{|k|\leq\infty} |\hat{f}_k| < \infty \rightarrow \text{then } ||f(x) - F_N||_{L^\infty_{[0,2\pi]}} \rightarrow 0 \quad (2)$$

Equation (1) and (2) are theorems in [5] regarding the convergence of the Series given that the coefficients $\hat{f}_k$ converges.

To ensure a faster rate of decay of $\hat{f}_k$ for discontinuous problems, we multiply by a function $\sigma(\frac{k}{N})$ in the *retrieval* stage. The requirement for filter functions can be found in [3, 1]

$$F_N^\sigma(x) = \sum_{k=-N}^{N} \sigma(\frac{k}{N})\hat{f}_k e^{ikx}$$

We explore different filtering function and compare the result.



(a) (b)

Figure 5: Parts (a) $f(x) = x$ (b)log scale error of $f(x) = sin(x)$

Fourier-Lagrange Hybrid

We examined that a subinterval $[a, b] \subset [-1, 1]$ is free of the discontinuity and will not be affected by Gibbs Phenomenon [3]. To test the idea, we cut-off and disregarded any subinterval containing the discontinuity and see the error of the reconstruction when the phenomenon was not present. The results were very nice, much like in the continuous case. This new approximation was not on the entire interval $[-1, 1]$ but on a subinterval. We used polynomial approximation over the subinterval containing the function's discontinuity. This method is introduce to eliminate Gibbs Phenomenon in discontinuous function.

$$F(x) = \begin{cases} P_n(x) & : -1 \leq x < -0.8 \\ F_N(x) & : -0.8 \leq x \leq 0.8 \\ P_n(x) & : -0.8 \leq x < -1 \end{cases}$$

Our method removes the Gibbs Phenomenon and produces much better result for discontinuous/non-periodic approximations.

Conclusion/Future Work

Many of the techniques in [3, 4, 1] requires knowing the location of the discontinuity for best result. We will look at some *edge detection* techniques and integrate it with the different methods and compare the results. Localization of the discontinuity can be obtained from the coefficients $\hat{f}_k$ [3]. To accomplish this, we will look at the *Sobolev q-norm* denoted by $H_p^q[0, 2\pi]$, measure of the smoothness of the derivatives and the function:

$$||u||^2_{H_p^q[0,2\pi]} = \sum_{m=0}^{q} \int_0^{2\pi} |u^m(x)|^2 dx$$

We will explore this idea to improve the hybrid method above and analyze the error.

We also plan on projecting the Fourier reconstruction onto the Gegenbauer polynomial as done in [**?**]. We will also try to reproject the hybrid method, where no Gibbs Phenomenon exists, on the same polynomial and see what happens.

We will study the Aliasing-Error and see how each method affects it.

We plan on exploring Aliasing-Error and its convergence rate further in detail.

References

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