Hybrid Trigonometric Polynomial Approximation

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Abstract

Fourier Series approximations are well known for their spectral convergence of data reconstructions on smooth and periodic functions. However, they fail to produce similar convergence when faced with discontinuous problems due to peculiar behavior near the discontinuities. Our work is to remedy this problem by using a hybrid method. In this method, polynomial approximation is used near the discontinuity and Fourier approximations are used on the other regions. We present numerical differences between our methods and other previous methods applied to similar popular problems.

Introduction

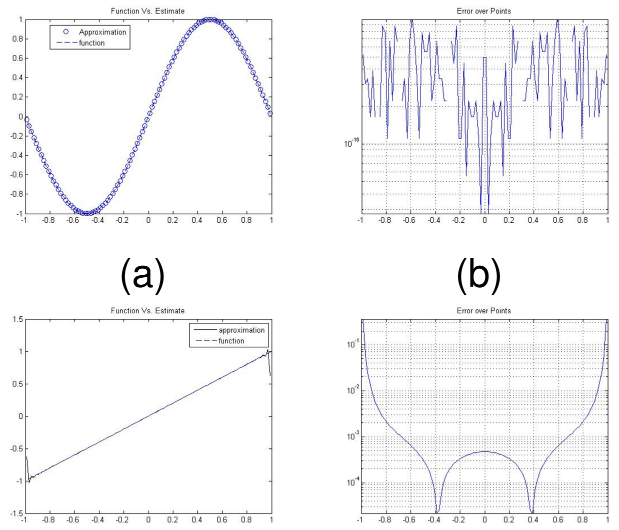
The goal of the project is to construct accurate point values of an unknown function $f(x)$ on $-1 \leq x \leq 1$. Given the first $2N+1$ coefficients, the Fourier Partial Sum

$$F_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(\pi k x) + b_k \sin(\pi k x)$$
$$a_k = \int_{-1}^{1} f(x)\cos(\pi k x)dx$$
$$b_k = \int_{-1}^{1} f(x)\sin(\pi k x)dx$$

Summing up the straight forward Fourier Series to construct an approximation is a good accurate reconstruction given that $f(x)$ is smooth and periodic:

$$max_{a \leq x \leq b} |f(x) - F_N(x)| \leq e^{-N}$$

the approximation converges spectrally, the error decays exponentially. However, the convergences rate for the approximation of discontinuous/non-periodic functions from the Fourier coefficients is very poor, as seen in [1].



(a) (b) (c) (d)

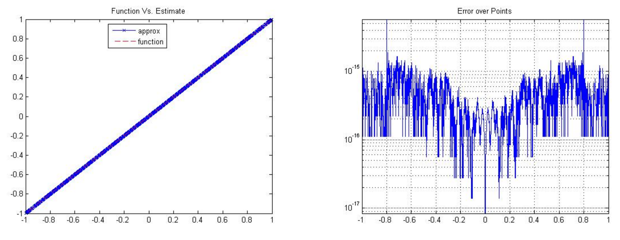
Figure 1: *Parts (a) approximation of* $f(x) = \sin(\pi x)$ *(b) the error plot, log scale, of the approximation of* $f(x) = \sin(\pi x)$ *(c)approximation of* $f(x) = x$ *(d) the error plot, log scale, of the approximation of* $f(x) = x$

This behavior is known as *Gibbs Phenomenon*. To obtain the same spectral convergence for discontinuous/non-periodic functions we aim to remove *Gibbs Phenomenon*.

Fourier-Lagrange Hybrid

We examined that a subinterval $[a, b] \subset [-1, 1]$ is free of the discontinuity and will not be affected by Gibbs Phenomenon [1]. To test the idea, we cut-off and disregarded any subinterval containing the discontinuity and see the error of the reconstruction when the phenomenon was not present. The results were very nice, much like in the continuous case. This new approximation was not on the entire interval $[-1, 1]$ but on a subinterval. We used polynomial approximation over the subinterval containing the function's discontinuity. This method is introduce to eliminate Gibbs Phenomenon in discontinuous function.

$$F(x) = \begin{cases} P_n(x) & : -1 \leq x < -0.8 \\ F_N(x) & : -0.8 \leq x \leq 0.8 \\ P_n(x) & : -0.8 \leq x < -1 \end{cases}$$



(a) (b)

Figure 2: *Parts (a)* $f(x) = x$ *(b)log scale error of* $f(x) = sin(x)$

Our method removes the Gibbs Phenomenon and produces much better result for discontinuous/non-periodic approximations.

Window

Note that in this section, we use Fourier Series complex form

$$F_N(x) = \sum_{k=-N}^{N} \hat{f}_k e^{ikx}$$
$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$$

When the coefficient $\hat{f}_k$ is calculated using quadrature, its called collocation

$$\hat{f}_k = \frac{1}{NP}\sum_{j=0}^{Np-1} f(x_j)e^(-ikx_j)$$

Using the following idea:

$$f(x) = \frac{f(x) * w(x)}{w(x)}$$
$$F_N(x) \approx \sum_{k=-N}^{N} \hat{f}_k e^{ikx}$$

We multiply the given data points of the function $f(x)$ to be reconstructed by a continuous function on the domain $[0, 2\pi]$ as done in [6].

$$\hat{f^w_k} = \frac{1}{2\pi}\int_0^{2\pi} w(x) * f(x)e^{-ikx}dx$$
$$F^w_N(x) \approx \frac{1}{w(x)}\sum_{k=-N}^{N} \hat{f^w_k} e^{ikx}$$

The function $w(x)$ must be a continuous function satisfying the following condition

$$w(x) = \begin{cases} 1 : x \in [a, b] \subset [0, 2\pi] \\ \epsilon \quad\quad : x \notin [a, b] \end{cases}$$

We use ϵ to avoid division by 0. In practice, $\epsilon \approx eps$, machine precision. On $[0, 2\pi]$, we use

$$w(x) = e^{(-\frac{(x-\pi)^{2\lambda}}{\pi})}$$

Instead of just cutting off the subinterval $[a, b] \subset [0, 2\pi]$ as described in [1] this allows us to *nicely* dissect the interval and replace it with polynomial interpolation, just as done in [6] to obtain better result near the end of the interval.

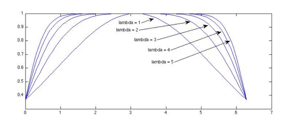
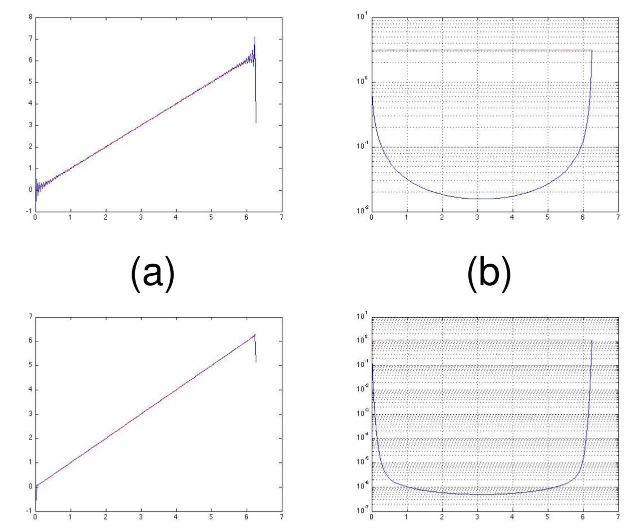


Figure 3: $w(x)$ *for different lamdba (*$\lambda = 1, 2, 3, 4, 5$*)*

The following figures are comparison between this new method and the straight forward Fourier Series



(a) (b) (c) (d)

Figure 4: *Parts (a)*$f(x) = x$ *with* $np = 201$ *and* $nc = 201$ *(b) the error plot, log scale,* $f(x) = x$ *with* $np = 201$ *and* $nc = 201$ *(c)*$f(x) = x$ *with* $np = 201$ *and* $nc = 201$ *(d) the error plot, log scale,* $f(x) = x$ *with* $np = 201$ *and* $nc = 201$

Filtering

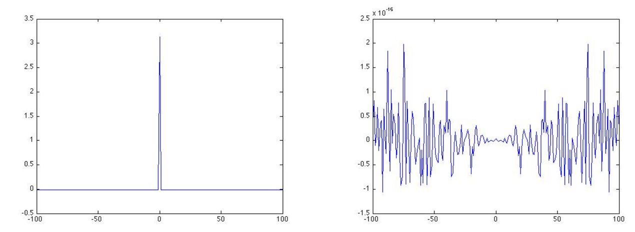
In this section, we use the Fourier Series complex form

There are two stages for providing information about a function [1]:

- *Storage*: store the expansion coefficients
- Retrieval: Sum up the expansion

For the case of the discontinuous problem, it is unwise to simply sum up the coefficients. This is because they contain data polluted by the discontinuities and this information will jeopardize the reconstruction.

As described in [1, 2, 3, 7, 8], the coefficient $\hat{f}_k$ decays at a faster rate for continuous/periodic problems than the cases where *Gibbs Phenomenon* is present.



(a) (b)

Figure 5: *Parts (a) coefficient of* $f(x) = x$ *(b) coefficient of* $f(x) = sin(x)$

To ensure a faster rate of decay of $\hat{f}_k$ for discontinuous problems, we multiply by a function $\sigma(\frac{k}{N})$ in the *retrieval* stage. The requirement for filter functions can be found in [1, 5, 7, 8]

$$F^{\sigma}_N(x) = \sum_{k=-N}^{N} \sigma(\frac{k}{N})\hat{f}_k e^{ikx}$$

We explore different filtering function and compare the result.

Conclusion/Future Work

Many of the techniques in [1, 2, 5, 7, 8] requires knowing the location of the discontinuity for best result. We will look at some *edge detection* techniques and integrate it with the different methods and compare the results. Localization of the discontinuity can be obtained from the coefficients $\hat{f}_k$ [1]. We will explore this idea to improve the hybrid method above and analyze the error.

We also plan on projecting the Fourier reconstruction onto the Gegenbauer polynomial as done in [**?**, 7]. We will also try to reproject the hybrid method, where no Gibbs Phenomenon exists, on the same polynomial and see what happens.

We will study the Aliasing-Error and see how each method affects it.

We plan on exploring Aliasing-Error and its convergence rate further in detail.

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