Central Binomial Coefficients

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Let $\omega(n,k)$ denote the number of distinct prime factors of $\binom{n}{k}$. Erdös [1, 2] proved that

$$\omega(2n,n) \sim 2\ln(2)\frac{n}{\ln(n)}$$

as $n \to \infty$ and wondered what else could be said about the prime factors. Let

$$f(n) = \sum_{\substack{p \le n, \\ p \nmid \binom{2n}{n}}} \frac{1}{p},$$

then [3, 4]

$$c = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n) = \sum_{k=2}^{\infty} \frac{\ln(k)}{2^k} = 0.5078339228...,$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} (f(n)-c)^2 = 0.$$

These two facts together express that $f(n) \to c$ for almost all integers n, hence $\binom{2n}{n}$ is almost always divisible by high powers of small primes. Let $g(n)$ be the smallest odd prime factor of $\binom{2n}{n}$. Whether $f(n)$ or $g(n)$ are bounded remains an open question.

Sárközy [5] and others [6, 7, 8] proved that $\binom{2n}{n}$ is not square-free for any $n > 4$. The largest n for which $\binom{2n}{n}$ is not divisible by p^2 for any odd prime p is $n = 786$.

We turn attention to $\binom{n}{k}$, the $(k+1)^{\text{st}}$ element in the n^{th} row of Pascal's triangle. For each $k \ge 1$, the sequence of integers n such that $\binom{n}{k}$ is square-free has asymptotic density c_k, where

$$c_1 = \frac{6}{\pi^2} = 0.6079271018..., \qquad c_2 = \frac{3}{4}\prod_{p\ge 3}\left(1-\frac{2}{p^2}\right) = 0.4839511484...$$

(the latter is related to the Feller-Tornier constant [9]). More generally, write k in base p:

$$k = a_0 + a_1 p + a_2 p^2 + \cdots + a_\ell p^\ell, \qquad 0 \le a_j < p \text{ for all } 0 \le j \le \ell, \qquad a_{\ell+1} = 0,$$

and define

$$c_{k,p}=\begin{cases}\prod\limits_{i=0}^{\ell}\left(1-\dfrac{a_i}{p}\right)\cdot\left(1+\sum\limits_{j=0}^{\ell}\dfrac{a_j(p-1-a_{j+1})}{(p-a_j)(p-a_{j+1})}\right) & \text{if } p\leq k,\\ 1-\dfrac{k}{p^2} & \text{if } p>k.\end{cases}$$

Then c_k is equal to $\prod_p c_{k,p}$, where the product is taken over all primes p. We have $c_3=0.251...$, $c_4=0.360...$, $c_5=0.191...$, $c_6=0.189...$, $c_7=0.062...$ and

$$0<c_k=\exp\left[-(\alpha+o(1))\sqrt{k}/\ln(k)\right]$$

as $k\to\infty$, where

$$\begin{aligned}\alpha &= \sum_{j=1}^{\infty}\frac{1}{j\,(j+1)}\int_0^{\infty}\{x\}^j x^{-3/2}dx\\ &= \sum_{j=1}^{\infty}\binom{2j}{j}\zeta(j+1/2)\frac{1}{2^{2j-1}}\left(1-j\sum_{i>j}\frac{1}{i^2}\right)\\ &= 1.825108....\end{aligned}$$

Integrals involving $\{x\}=x-\lfloor x\rfloor$ as such also appear in [10, 11]. It follows that there are $\sim\tau\,N$ square-free binomial coefficients $\binom{n}{k}$ with $0\leq k<n\leq N$, where

$$\tau=2\sum_{k=0}^{\infty}c_k=2(5.3275...)=10.655....$$

In words, each row of Pascal's triangle possesses approximately $10\frac{2}{3}$ square-free entries (on average).

0.1. Relevant Sums. Let φ denote the Golden mean $(1+\sqrt{5})/2$. We have [12, 13, 14, 15, 16]

$$\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}}=\frac{1}{3}+\frac{2\sqrt{3}\pi}{27},\qquad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\binom{2n}{n}}=\frac{1}{5}+\frac{4\sqrt{5}\ln(\varphi)}{25},$$

$$\sum_{n=1}^{\infty}\frac{n}{\binom{2n}{n}}=\frac{2}{3}+\frac{2\sqrt{3}\pi}{27},\qquad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{\binom{2n}{n}}=\frac{6}{25}+\frac{4\sqrt{5}\ln(\varphi)}{125},$$

$$\sum_{n=1}^{\infty}\frac{n^2}{\binom{2n}{n}}=\frac{4}{3}+\frac{10\sqrt{3}\pi}{81},\qquad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2}{\binom{2n}{n}}=\frac{4}{25}-\frac{4\sqrt{5}\ln(\varphi)}{125},$$

$$\sum_{n=1}^{\infty} \frac{n^3}{\binom{2n}{n}} = \frac{10}{3} + \frac{74\sqrt{3}\pi}{243}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{\binom{2n}{n}} = -\frac{2}{125} - \frac{28\sqrt{5}\ln(\varphi)}{625}$$

and, more generally [17],

$$\sum_{n=1}^{\infty} \frac{n^k}{\binom{2n}{n}} = p_k + q_k\sqrt{3}\pi, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^k}{\binom{2n}{n}} = r_k + s_k\sqrt{5}\ln(\varphi)$$

for appropriate rationals p_k, q_k, r_k, s_k. Let L_D denote the Dirichlet L-series with character $(D/\cdot)$ and Li_k denote the k^{th} polylogarithm function. The following are more difficult [12, 13, 14, 15, 16]:

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n} = \frac{\sqrt{3}\pi}{9}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n} n} = \frac{2\sqrt{5}\ln(\varphi)}{5},$$

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^2} = \frac{\pi^2}{18}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n} n^2} = 2\ln(\varphi)^2,$$

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^3} = \frac{\sqrt{3}\pi}{2} L_{-3}(2) - \frac{4\zeta(3)}{3}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n} n^3} = \frac{2\zeta(3)}{5},$$

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^4} = \frac{17\pi^4}{3240},$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n} n^4} \quad = \quad & 8\,\mathrm{Li}_4\left(\frac{1}{\varphi}\right) + 8\ln(\varphi)\,\mathrm{Li}_3\left(\frac{1}{\varphi}\right) - \frac{1}{2}\,\mathrm{Li}_4\left(\frac{1}{\varphi^2}\right) \\ & + \frac{7\pi^2\ln(\varphi)^2}{15} - \frac{13\ln(\varphi)^4}{6} - \frac{4\zeta(3)\ln(\varphi)}{5} - \frac{7\pi^4}{90},\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^5} = \frac{9\sqrt{3}\pi}{8} L_{-3}(4) + \frac{\pi^2\zeta(3)}{9} - \frac{19\zeta(5)}{3},$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n} n^5} \quad = \quad & \frac{5}{2}\,\mathrm{Li}_5\left(\frac{1}{\varphi^2}\right) + 5\ln(\varphi)\,\mathrm{Li}_4\left(\frac{1}{\varphi^2}\right) \\ & + 4\zeta(3)\ln(\varphi)^2 - \frac{4\pi^2\ln(\varphi)^3}{9} + \frac{4\ln(\varphi)^5}{3} - 2\zeta(5).\end{aligned}$$

We wonder whether the last two alternating series possess expressions involving L-series values rather than polylogarithmic values. Let $G = L_{-4}(2)$ denote Catalan's constant. Other series include

$$\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)} = \frac{2\sqrt{3}\pi}{9}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n}(2n+1)} = \frac{4\sqrt{5}\ln(\varphi)}{5},$$

$$\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2} = \frac{8G}{3} - \frac{\pi\ln(2+\sqrt{3})}{3}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n}(2n+1)^2} = \frac{\pi^2}{6} - 3\ln(\varphi)^2$$

and

$$\sum_{n=0}^{\infty} \frac{2^n}{\binom{2n}{n}(2n+1)} = \frac{\pi}{2}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{\binom{2n}{n}(2n+1)} = \frac{2}{\sqrt{3}} \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right),$$

$$\sum_{n=0}^{\infty} \frac{2^n}{\binom{2n}{n}(2n+1)^2} = 2L_{-8}(2) - \frac{\sqrt{2}\pi}{4} \ln(1+\sqrt{2}),$$

$$\sum_{n=0}^{\infty} \frac{2^{2n}}{\binom{2n}{n}(2n+1)^2} = 2G, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{\binom{2n}{n}(2n+1)^2} = \frac{\pi^2}{8} - \frac{1}{2}\ln(1+\sqrt{2})^2,$$

but similar expressions for

$$\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^3}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n}(2n+1)^3}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{\binom{2n}{n}(2n+1)^2}$$

remain open (as far as is known).

Batir [18, 19] recently proved that

$$\sum_{n=1}^{\infty} \frac{2^{4n}}{\binom{2n}{n}^2 n^3} = 8\pi G - 14\zeta(3), \qquad \sum_{n=0}^{\infty} \frac{2^{4n+2}}{\binom{2n}{n}^2 (2n+1)^3} = 14\zeta(3) - 4\pi G$$

and also derived a complicated formula for $\sum_{n=1}^{\infty} 1/\binom{3n}{n}$. We will barely mention cases for which $\binom{2n}{n}$ is in the numerator, for example [12, 14, 20],

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{2n}(2n+1)} = \frac{\pi}{2},$$

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{2n}(2n+1)^2} = \frac{\pi\ln(2)}{2}, \qquad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{3n}(2n+1)^2} = \frac{\sqrt{2}}{8}\left(\pi\ln(2) + 4G\right),$$

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{4n}(2n+1)^2} = \frac{3\sqrt{3}}{4} L_{-3}(2),$$

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{4n}(2n+1)^3} = \frac{7\pi^3}{216}, \qquad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{4n}(2n+1)^4} = \frac{\pi\zeta(3)}{12} + \frac{27\sqrt{3}}{64} L_{-3}(4),$$

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{1}{2^{4n}(2n+1)} = \frac{4G}{\pi}.$$

Similar expressions for

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{3n}(2n+1)^3}, \qquad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{3n}(2n+1)^4}, \qquad \sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{(-1)^n}{2^{4n}(2n+1)}$$

remain open (as far as is known). Techniques in [21] might be helpful in evaluating sums as these.

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